EXACT ANALYTICAL SOLUTION FOR OBLIQUE INCIDENCE ON A GRADED INDEX INTERFACE BETWEEN A RIGHT-HANDED AND A LEFT-HANDED MATERIAL

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We have determined the scattering parameters of optical structures incorporating both left-handed metamaterials (LHM) and conventional ("right-handed") materials (RHM) for the case when the refractive index at the LHM-RHM interface is graded, for oblique incidence at an arbitrary angle and for arbitrary spectral dispersion. We derived an accurate analytical solution to Helmholtz' equation for the case of the refractive index gradient varying as a hyperbolic tangent where the steepness of the index transition may be arbitrary, even of the order of vacuum wavelength. We determined the expressions for the field intensity along the LHM-RHM structure with refractive index gradient. We show that there is an excellent agreement between our analytical results and the accurate numerical simulations done by finite element method.

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1. Introduction

Electromagnetic metamaterials are a novel class of artificial composites with properties not readily found in Nature. Probably the best known metamaterials are those with negative effective refractive index, called left-handed metamaterials (LHM). It was shown theoretically by Veselago [1] that left-handed materials exhibit numerous remarkable properties, which include e.g. negative refractive index, reversal of Snell's law, inverse Doppler effect and Goos-Hänchen shift, radiation tension instead of pressure, etc. All these exotic and sometimes counter-intuitive properties are the result of the fact that the Poynting vector in these materials is antiparallel to the wavevector. In other words, the electric field, the magnetic field and the wavevector of a plane electromagnetic wave form a left-handed system of reference – whence the name of such structures. The LHM are structured at the subwavelength level and most of their implementations include arrays of electromagnetic resonators or "particles" which simultaneously furnish negative effective dielectric permittivity and negative magnetic permeability. Examples of the LHM particles include double split-ring resonators and nanowires, cut wire pairs, etc. [2].

The concept of negative refractive index metamaterials was brought to the attention of the scientific community and to the practical implementation by the works by Pendry [3, 4]. He suggested double split-ring resonators and wire arrays as the first metamaterial particles. Shelby et al published the first experimental demonstration of a left-handed material in 2001 [5]. Besides the combination of the split-ring resonators and wire arrays as proposed by Pendry [4], various other LHM particles were introduced for the optical range, including coupled split-ring resonators [6] cut-wire pairs and fishnet structures [1]. Of those, the best performance and the highest operating frequencies to date were obtained by fishnet structures. Among the main challenges in the field of

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optical LHM are the reduction of large losses, spreading of narrow ranges of negative index and avoidance of parasitic effects like ballistic inductance. Numerous applications of optical metamaterials have been proposed. These include superlenses that enable near-field imaging below the diffraction limit [7, 8], hyperlenses for far-field sub-diffraction imaging [9], waveguides that can stop light [10], miniaturized photonic devices such as subwavelength Fabry-Perot resonators and waveguides [11, 12], even invisibility cloaks utilizing the technique of transformation optics [13, 14].

Inhomogeneous electromagnetic metamaterials with refractive index gradient are of great interest for graded metamaterial lenses and for the transformation optics in general. For instance, they could be very important for the hyperlenses which offer possibilities to perform optical analysis of biomaterials at nanometric level [9]. The introduction of the index gradient provides an additional degree of freedom in the design of such structures. The works studying NRM-containing structures with gradient refractive index include [15-17]. An analytical solution for some special cases of graded NRM interfaces was presented in [18]. An experimental demonstration of gradient NRM lens was published by Smith et al [19]. Reflections within graded NRM structures were taken in account in [20], and a solution for normal incidence was proposed in [21].

In the present paper, we consider the oblique incidence of the electromagnetic waves and the transmission and reflection properties of optical structures with a graded transition from a right-handed to a left-handed material or vice versa. An exact analytical solution of Helmholtz' equation is given for the case when both effective dielectric permittivity and magnetic permeability vary according to a hyperbolic tangent law where the steepness of the interface may be arbitrary. We present an analytical expressions for the reflection and transmission coefficients of such structures as well as the spatial distribution of the electromagnetic field in them. At the end we compare our analytical solution for various incidence angles with the results obtained by accurate numerical simulations based on a finite element method.

2. Field equation

In our derivation we assume that the material optical properties can be described by its effective dielectric permittivity and the effective magnetic permeability, i.e. that we can introduce the effective refractive index. This is an approach customarily used for metamaterials, since their 'particles' have subwavelength dimensions. We consider the situation where the effective parameters change along one direction only (x-axis). We assume that the electric and magnetic field vectors are given by

$$\begin{aligned} \vec{E}(\vec{r}) &= -E(x,y)\,\vec{z}_0 \ , \\ \vec{H}(\vec{r}) &= H(x,y)\cos\theta\,\vec{y}_0 - H(x,y)\sin\theta\,\vec{x}_0 \ , \end{aligned} \tag{1}$$

where θ is the incident angle. The wave equation for the electric field is then obtained from the Maxwell equations in the form

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} - \frac{1}{\mu}\frac{d\mu}{dx}\frac{\partial E}{\partial x} + \omega^2 \mu\varepsilon E(x,y) = 0 \tag{2}$$

where $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega, x)$ are the frequency- and space-dependent effective dielectric permittivity and magnetic permeability, respectively. This equation describes the electromagnetic wave propagation through a medium with the constitutive parameters varying along the propagation direction. The spatial dependency of the functions $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega, x)$ may be completely arbitrary, even on space scales shorter than the wavelength of the radiation, under the obvious condition that the effective medium approximation remains valid.

We choose the hyperbolic tangent tanh x as the most convenient function to describe the spatial dependence of $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega, x)$, since it provides correct asymptotic values in both materials and allows a detailed study of both the limit of the abrupt transitions and the slowly varying interface. There are no restrictions regarding the functions $\mu_{eff}(\omega)$ and $\varepsilon_{eff}(\omega)$ – the approach is applicable to arbitrary spectral dispersions.

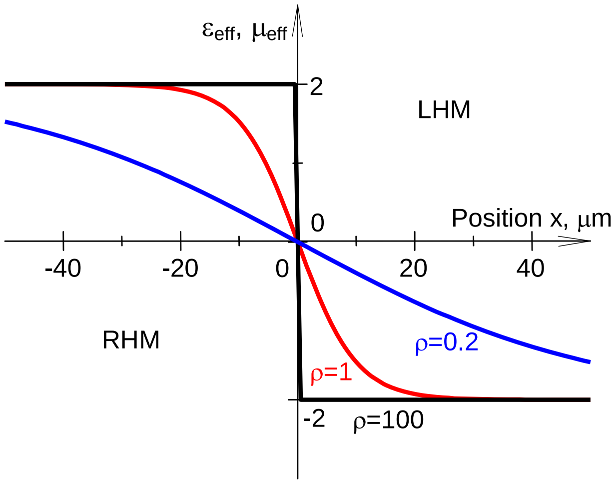

Fig. 1. We assume in this paper that both the effective permittivity ε_{eff} and the permeability μ_{eff} vary along the propagation direction according to a hyperbolic tangent function, parameter ρ determining its abruptness

The effective permittivity and permeability are given by

$$\mu = -\mu_0 \mu_R(\omega) \tanh(\rho x),$$
$$\varepsilon = -\varepsilon_0 \varepsilon_R(\omega)\left[1 + i\frac{\sigma(\omega)}{\omega\varepsilon_0\varepsilon_R(\omega)}\right]\tanh(\rho x), \quad (3)$$

where ρ is a parameter describing the abruptness of the transition from the right-handed material (RHM, to the left of the plane $x = 0$) to the left-handed material (LHM, to the right of the plane $x = 0$). The dispersive functions $\varepsilon_R(\omega)$, $\sigma(\omega)$ and $\mu_R(\omega)$ are the relative permittivity, conductivity and relative permeability of the media, respectively. The impedance

$$Z = Z_0 Z(\omega) = \sqrt{\mu_0 \mu_{eff}(\omega) / \varepsilon_0 \varepsilon_{eff}(\omega)}$$

is constant throughout the entire structure; as a result, there is no reflection on the graded interface between the two materials.

In order to solve Eq. (2), we assume that the electric field strength can be written in the form $E(x, y) = X(x)Y(y)$. Thus we obtain two ordinary differential equations for the functions $X(x)$ and $Y(y)$ in the form

$$\frac{d^2X}{dx^2} - \frac{1}{\mu}\frac{d\mu}{dx}\frac{dX}{dx} + \left(\omega^2\mu\varepsilon - k_y^2\right)X(x) = 0$$
$$\frac{d^2Y}{dy^2} + k_y^2 Y(y) = 0, \quad (4)$$

where $k_y = k \sin\theta$. The second equation in (4) is elementary and its solution is $Y(y) = \exp(\pm k_y y)$. The standard approach to the solution of the first equation in (4) is to eliminate the first order terms by introducing the function $F(x)$ instead of the function $X(x)$ using the following transformation

$$X(x) = \sqrt{\mu(x)}F(x) \quad . \tag{5}$$

In this way, we obtain the following wave equation for the function $F(x)$

$$\frac{d^2F}{dx^2} + \left[\omega^2\mu\varepsilon + \frac{1}{2\mu}\frac{d^2\mu}{dx^2} - \frac{3}{4\mu^2}\left(\frac{d\mu}{dx}\right)^2\right]F(x) = 0 \quad . \tag{6}$$

This equation can also be written as a wave equation

$$\frac{d^2F}{dx^2} + k_\mu^2(x)F(x) = 0 \quad , \tag{7}$$

where

$$k_\mu^2(x) = \omega^2\mu\varepsilon + \frac{1}{2\mu}\frac{d^2\mu}{dx^2} - \frac{3}{4\mu^2}\left(\frac{d\mu}{dx}\right)^2 \tag{8}$$

is the space-dependent effective wave vector for the electric field. In case of the hyperbolic tangent profile for the functions $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega, x)$, Eq. (7) is generally reduced to a hypergeometric equation, allowing for analytical solution in terms of suitable hypergeometric functions.

3. Analytical solutions of the field equation

As mentioned previously, in this paper we consider an inhomogeneous medium consisting from a conventional material part (RHM) and a left-handed metamaterial part (LHM) between which the effective permittivity and permeability vary in space according to a hyperbolic tangent function. The exact analytical solution of Eq. (2), with the proper normalization, is given by

$$\begin{aligned} E(x) = E_0 & \frac{\Gamma\left(1 - i\frac{\kappa}{2\rho}(\cos\theta + 1)\right)\Gamma\left(1 - i\frac{\kappa}{2\rho}(\cos\theta - 1)\right)}{\Gamma(2)\Gamma(-i(\kappa/\rho)\cos\theta)} \\ & \times\left[e^{\rho x} + \varepsilon^{-\rho x}\right]^{-i(\kappa/\rho)\cos\theta}\tanh^2\rho x \\ & \times {}_2F_1\left(1 + i\frac{\kappa}{2\rho}(\cos\theta + 1),\ 1 + i\frac{\kappa}{2\rho}(\cos\theta - 1),\ 2;\ \tanh^2\rho x\right) \\ & \times e^{i\kappa y \sin\theta} \end{aligned} \quad ,(9)$$

where ${}_2F_1(a\ ,\ b\ ,\ c\ ;\ z)$ is the hypergeometric function, E_0 is the amplitude of the electric field of the incident electromagnetic wave far to the left from the interface between the two materials, and

$$
\begin{aligned}
&\kappa^2 = -\omega^2 \varepsilon_{eff}(\omega)\mu_{eff}(\omega),\\
&\kappa = k + i\alpha,\\
&k = \mathrm{Re}(\kappa),\\
&\alpha = \mathrm{Im}(\kappa)
\end{aligned}
\tag{10}
$$

For lossless media (σ→0 or κ = *k*) we obtain the asymptotic expression for the electric field *E*(*x*) in the limits $x \to \pm\infty$, as

$$
\begin{aligned}
E(\mp\infty) &= E \quad \exp\left(\pm i\vec{k}\vec{r}\right) =\\
&= E_0 \exp\left[\pm ik(x\cos\theta + y\sin\theta)\right]
\end{aligned}
\tag{11}
$$

From the asymptotic expression (11) we see that to the left of the interface at *x* = 0, i.e. in the right-handed material (ε > 0, μ > 0), we have an electromagnetic wave with the wave vector $\vec{k}_1 = +\vec{k}$ propagating to the right. On the other side, to the right of the interface at *x* = 0, i.e. in the left-handed material (ε < 0, μ < 0), we have an electromagnetic wave with the wave vector $\vec{k}_2 = -\vec{k}$ propagating to the right without reflection. This exact solution is valid for arbitrary steepness ρ of the graded index interface.

In the special case of the normal incidence (θ = 0), the exact analytical solution acquires a remarkably simple form

$$
E(x) = E_0 \exp\left(-i\frac{\kappa}{\rho}\ln 2\right)\left[\cosh(\rho x)\right]^{-i\frac{\kappa}{\rho}} \quad , \tag{12}
$$

For lossless media (σ→0 or κ = *k*) we obtain the asymptotic expression for the electric field *E*(*x*) in the limits $x \to \pm\infty$, as

$$
E(\mp\infty) = E_0 \exp(\pm ikx) \quad . \tag{13}
$$

From Eq. (13) we see that the wave described by it is an electromagnetic wave with a wavevector $\vec{k}_{-\infty} = +k\,\vec{e}_x$ in the right-handed material far from the interface ($x \to -\infty$). This is a wave that propagates in the +*x* direction, i.e., a wave propagating to the right. On the other hand, for $x \to +\infty$, we have an electromagnetic wave with wave vector $\vec{k}_{+\infty} = -k\,\vec{e}_x$; this represents a wave of which the phase fronts propagate in the −*x* direction.

However, since we have a left-handed material for *x* > 0, the energy flux (Poynting vector) is still propagating from left to right. This is perfectly consistent with the fact that there is no reflection on this structure.

If we instead assume that the amplitude of the normal electric field in the right-handed material far from the interface $x \to -\infty$ denoted by E_0 is a complex number with the phase $\kappa/\rho \ln 2$, the simple analytical result Eq. (13) in the special case of the normal incidence becomes

$$
E(x) = E_0 \left[\cosh(\rho x)\right]^{-i\frac{\kappa}{\rho}} \tag{14}
$$

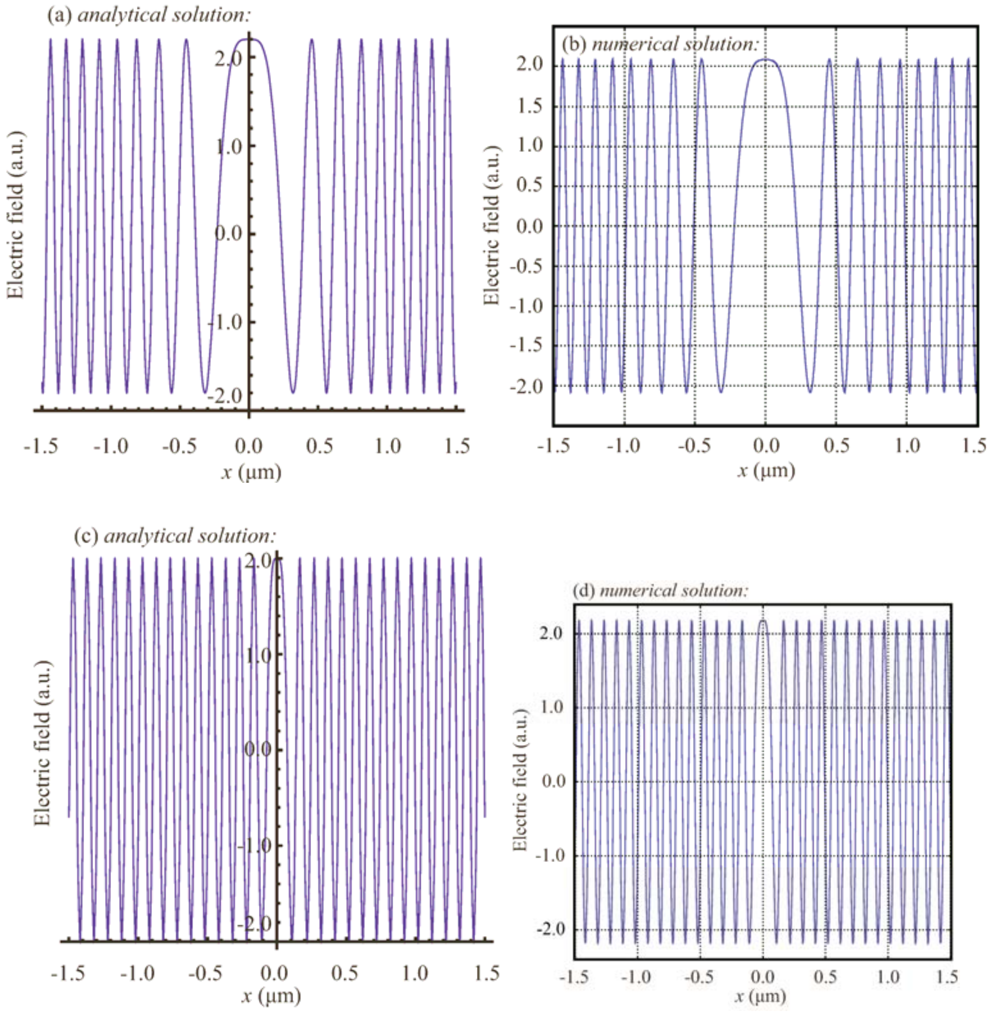

Fig. 2. Comparison of the analytical and the numerical results for the transmitted electric field. We plot E(x) at t = 0 as a function of x. (a) Analytical solution for $\rho = 10\ \mu m^{-1}$. (b) Numerical solution for $\rho = 10\ \mu m^{-1}$. (c) Analytical solution for $\rho = 1\ \mu m^{-1}$. (d) Numerical solution for $\rho = 1\ \mu m^{-1}$.

4. Comparison with numerical results

In order to validate our exact analytical solution in the special case of the normal incidence, we compared the waveforms with results obtained from a direct simulation of Maxwell's equations. We used a finite element method (COMSOL Multiphysics) for their solution. We utilized perfectly matched layers at the both sides of the structure to close the simulation domain. For the simulation results shown in Figs. 2(b) and (d), we used the following parameters: $\lambda_0 = 1$ μm and $\varepsilon_{eff}(\lambda_0) = \mu_{eff}(\lambda_0) = 1$. Figs. 2(a)-(b) and Figs. 2(c)-(d) are plotted for different degrees of the transition steepness. We see that there is an excellent agreement between the analytical and numerical results for the normal incidence.

The analytical results for the electric field strength E(x, y, t) for t = 0 and y = 0, in two cases of oblique incidence with the incident angles equal to $\theta = \pi/6$ and $\theta = \pi/4$ are shown in Figs. 3(a)-(b) respectively.

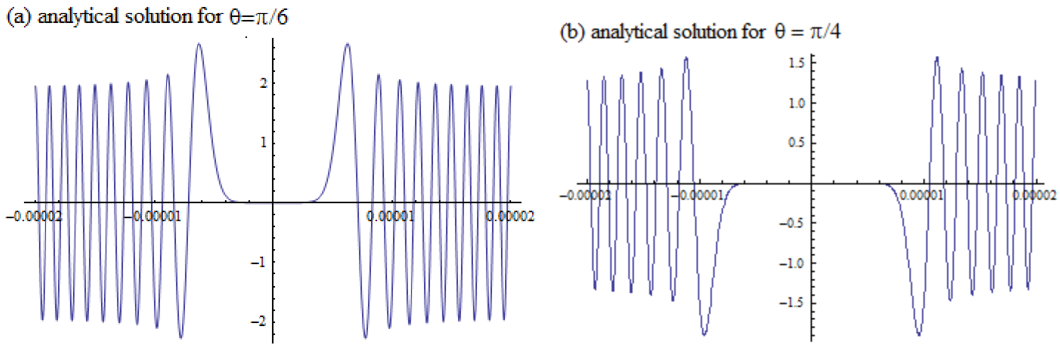

Fig. 3. Analytical results for the electric field. We plot $E(x, y)$ at $t = 0$ for $y = 0$ as a function of x. (a) Analytical solution for $\rho = 10\ \mu m^{-1}$ and $\theta = \pi/6$. (b) Analytical solution for $\rho = 10\ \mu m^{-1}$ and $\theta = \pi/4$.

5. Conclusion

We have investigated electromagnetic wave propagation with oblique incidence through a graded index interface between a right-handed and a left-handed material. We derived an exact analytic solution for the electric field strength when the refractive index varies in space as a hyperbolic tangent. The solution is valid for arbitrary steepness of the index transition, even when the effective constitutive parameters vary on the scale of the vacuum wavelength where the traditional approximate methods (e.g. the Slowly Varying Envelope Approximation) cease to work. We have validated our analytical solutions by accurate numerical simulations using a finite element method for both normal and oblique incidence. Further numerical studies in more general cases will be the subject of coming papers.

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